Libor Market Model
Version 1.0

1 Introduction

This plug-in implements the Libor Market Model (also know as BGM Model, from the authors Brace Gatarek Musiela). For a general reference on this model see [1], [2] and [3].

2 How to use the plug-in

In the Fairmat user interface when you create a new stochastic process you will find the additional option "Libor Market Model".

Due to the complexity of the Libor Market model, it's not possible to reduce the specification of the model to the setting of few parameters, so you have to proceed in this way:

1. first build an xml file with all the model data as described in section 4
2. insert a new Libor Market Model process selecting it in the process list
3. once a Libor Market Model process is inserted, right click on the process name and choose "import the model parameters from an xml file"
4. select the xml file you built at step 1.

3 Implementation Details

3.1 Notation and preliminaries consideration

We will use the following notations and definitions.

- $T_{-1}, T_0, T_1, \dots, T_N$ set of strictly increasing times, with $T_{-1} = 0$
- tenors are $\alpha_i = T_i - T_{i-1}$ for $i = 1, \dots, N$ (usually tenors are all equal)
- $p_i(t) = p(t, T_i)$ price in t of the zero coupon bond which pays 1 in T_i
- the forward libor rate at time t for the period $[T_{i-1}, T_i]$ is given by

$$L_i(t) = \frac{1}{\alpha_i} \frac{p_{i-1}(t) - p_i(t)}{p_i(t)} \quad , \quad i = 1, \dots, N \tag{1}$$

- we will indicate with Q^i the T_i-forward measure.[1]
- a cap contract with cap rate R and resettlement dates $T_0, T_1, \ldots, T_N$ is a contract that at every time T_i for $i = 1, \ldots, N$ pays the holder the amount

  $$X_i = \alpha_i \cdot \max[L_i(T_i - 1) - R, 0]. \tag{2}$$

  Note that the forward rate $L_i(T_i - 1)$ is the spot rate in T_{i-1} with maturity T_i and that the amount paid at time T_i is determined at time T_{i-1}. Each payment is called caplet.

The main goal of the Libor Market model is that of defining a non arbitrage model in which cap prices are given by Black-76 formula which states that the price of the X_i caplet is

$$Capl_i^B(t) = \alpha_i p_i(t)[L_i(t)N(d_1) - RN(d_2)] \tag{3}$$

where

$$d_1 = \frac{1}{\sigma_i \sqrt{T_{i-1} - t}} \left[\ln\left(\frac{L_i(t)}{R}\right) + \frac{1}{2}\sigma_i^2(T_{i-1} - t) \right] \tag{4}$$

$$d_2 = d_1 - \sigma_i \sqrt{T_{i-1} - t} \tag{5}$$

and where σ_i is called Black volatility for the caplet i.

3.2 Defining the model

Given

- W^N a k-dimensional Wiener process in the measure Q^N
- $\sigma_1(t), \ldots, \sigma_N(t)$ deterministic functions $\mathbb{R} \to \mathbb{R}^k$
- an initial term structure of non negative forward rates $L_1(0), \ldots, L_N(0)$

and calling W^i the Wiener process in the Q^i measure obtained with the Girsanov transformation $Q^N \to Q^i$, the Libor Market model is then defined assuming an evolution for the libor rates given by

$$dL_i(t) = L_i(t)\sigma_i(t)dW^i \quad , \quad i = 1, \ldots, N \tag{6}$$

3.3 Discrete savings account and the spot libor measure

To price derivatives is fundamental to express the evolution of the various libor rates under the same measure. In the short rate models framework is common to use the risk neutral measure. This is defined as the martingale measure

[1] The T-forward measure is the martingale measure that has the price of the zero coupon bond maturing at time T as numeraire

with the bank account as numeraire, and the bank account is defined using the instantaneous rate.

In the Libor Market model framework there's no such a thing as the short rate. It is common then to define a process called *discrete savings account* which represents the investment of a unitary amount at time T_0 in a T_1-bond, with a reinvestment at time T_2 in a T_3 bond, and so on. The process is thus described by

$$B(T_0) = 1 \tag{7}$$

$$B(T_n) = \prod_{k=0}^{n-1} \frac{1}{p(T_k, T_{k+1})} = \prod_{k=0}^{n-1} \left[1 + \alpha_{k+1} L(T_k, T_{k+1})\right]. \tag{8}$$

The measure that has B as numeraire will be denoted with Q^B and called spot libor measure. In the measure Q^B the dynamic of libor rates is given by

$$dL_i(t) = L_i(t) \left[\sum_{k=\beta(t)}^{i} \frac{\alpha_k L_k(t)}{1 + \alpha_k L_k(t)} \sigma_k(t) \sigma_i'(t) \right] dt + L_i(t) \sigma_i(t) dW^Q(t) \tag{9}$$

where $\beta(t)$ is the index of the first libor rate that will become a spot rate, more explicitly

$$\beta(t) = m \text{ if } t \in (T_{m-2}, T_{m-1}]. \tag{10}$$

3.4 Properties of the model

In the Q^i measure the dynamic is ruled by equation (6) and can be easily solved to give

$$L_i(T) = L_i(t) e^{\int_t^T \sigma_i(s) dW^i(s) - \frac{1}{2} \int_t^T \|\sigma_i(s)\|^2 ds}. \tag{11}$$

$L_i(T)$ is thus log-normal distributed, more precisely

$$L_i(T) = L_i(t) e^{Y_i(t,T)} \tag{12}$$

where Y_i is a normal distribution with mean

$$m(t,T) = -\frac{1}{2} \int_t^T \|\sigma_i(s)\|^2 ds \tag{13}$$

and variance

$$\Sigma(t,T)^2 = \int_t^T \|\sigma_i(s)\|^2 ds. \tag{14}$$

From this equations it follows that the price of a caplet is given by the Black-76 formula with Black volatility equal to

$$\sigma_i = \frac{1}{\sqrt{T_{i-1} - t}} \Sigma(t, T_{i-1}) = \sqrt{\frac{1}{T_{i-1} - t} \int_t^{T_{i-1}} \|\sigma_i(s)\|^2 ds}. \tag{15}$$

3.5 The choice of a functional form for the σ_i

Usually a model is calibrated imposing that prices of some instrument calculated through the model are as near as possible to prices of the same instrument seen in the market.

In the Libor Market model calibrating means *fixing the $\sigma_i(t)$ functions*, and due to the fact that Black-76 formula can be used to price caps, it is natural to fix this functions in order to recover cap prices taken from the market.

Of course, given a set of cap prices, there are infinite ways in which you can fix the σ_i to recover exactly that prices, so it's common to choose a particular functional form for the σ_i in order to fix them in a unique way.

In Fairmat you can choose between two type of specification for the σ_i

piecewise constant: it is assumed that each σ_i is constant in each interval $(T_i, T_{i+1}]$, thus

$$\sigma_i(t) = \sigma_{ij} \text{ for } t \in (T_{j-1}, T_j], i = 1, \dots, N \text{ and } j = 0, \dots, i-1 \tag{16}$$

In this case you have to specify the $N(N+1)/2$ values of the σ_{ij} to fix the model.

parametric: it is assumed that each σ_i has the form

$$\sigma_i(t) = \Phi_i \left[[a + b(T_{i-1} - t)] \, e^{-c(T_{i-1}-t)} + d \right] \tag{17}$$

In this case you have to specify N values for the Φ_i and the values for a, b, c, d

3.6 Simulation and discretization scheme

Fairmat simulates the dynamic of libor rates in the spot libor measure. Applying a direct Euler discretization to equation (9) one obtains

$$L_i(t+\Delta t) = L_i(t) - L_i(t) \left[\sum_{k=i+1}^{N} \frac{\alpha_k L_k(t)}{1 + \alpha_k L_k(t)} \sigma_k(t) \sigma_i'(t) \right] \Delta t + \\ + L_i(t) \sigma_i(t) \sqrt{\Delta t} N(0,1) \tag{18}$$

but the numerical stability can be improved by simulating the $\ln(L_i)$ processes through the formula

$$\ln\left(L_i(t+\Delta t)\right) = \ln(L_i(t)) - \left[\frac{1}{2}\sigma_i^2(t) + \sum_{k=i+1}^{N} \frac{\alpha_k L_k(t)}{1 + \alpha_k L_k(t)} \sigma_k(t) \sigma_i'(t) \right] \Delta t + \\ + \sigma_i(t) \sqrt{\Delta t} N(0,1). \tag{19}$$

which can be rewritten as

$$L_i(t+\Delta t) = L_i(t)\exp\left\{-\left[\frac{1}{2}\sigma_i^2(t) + \sum_{k=i+1}^{N} \frac{\alpha_k L_k(t)}{1+\alpha_k L_k(t)}\sigma_k(t)\sigma_i'(t)\right]\Delta t + \right. \\ \left. +\sigma_i(t)\sqrt{\Delta t}N(0,1)\right\}. \quad (20)$$

Fairmat uses this last formula to simulate libor rates.

4 How to build the xml files

This files contain informations which describe an instance of the Libor Market model, you need to create them in order to use the Libor Market model in Fairmat.

For vector fields different component values are separated by a tab character. For matrix fields each row is built as a vector and different rows are separated by a carriage return and a line feed (standard end of line in windows systems).

Maturities are measured in year fraction, rates and volatilities are written with their correct value without the % symbols (a 3.2% rate is written as 0.032 and a 15.2% volatility is written as 0.152).

The main tag is *LiborMM_ModelData* and the other fields contained in the files are the following:

Market a string describing the market to which data refer. Possible choices are:

- EU for Europe
- US for USA
- UK for United Kingdom
- SW for Switzerland
- GB for United Kingdom
- JP for Japan

Date is the date to which data refer, the format is ddmmyyyy

NumMaturities is the number of maturities specified for the zero rate curve

NumMaturVola is an integer representing N, the number of libor rates to be simulated

nfactors is the number of factors of the model

CalcVolaFun is a number representing the type of volatility specification. Possible choices are:

- 1 for the parametric specification

- 2 for the piecewise constant specification

Parms a vector representing the values for a, b, c, d of the parametric form specification as in equation (17)

Weight a vector representing Φ_i values of the parametric form specification as in equation (17)

ZR vector with zero coupon rates (to be used as continuously compounded rates, so that the price of the zero coupon bond maturing at time T is given by $e^{-ZR(T)\cdot T}$

ZRMarketDates vector of maturities corresponding to ZRMarket

tenor a scalar with the value of the tenor α supposed constant

v1,v2,v3 matrices in which the element ij specify the values to give at the first, second and third components of volatility σ_i in the interval $(T_i - 1, T_i)$

References

[1] Damiano Brigo and Fabio Mercurio. *Interest Rate Models, Theory and Practice.* Springer, second edition, 2006.

[2] Dariusz Gatarek. *The LIBOR Market Model in Practice.* Wiley, 2007.

[3] Tomas Björk. *Arbitrage theory in continuous time.* Oxford University Press, 2004.