Dai Singleton Model
Version 1.0

1 Introduction

This plug-in implements Dai-Singleton affine term structure models with 1,2,3 factors and also their extensions known as essentially affine models. For general references on affine models see [1] for completely affine models and [2] for essentially affine ones.

2 How to use the plug-in

- In the Fairmat user interface when you create a new stochastic process you will find the additional option "Dai-Singleton interest rate model".
- The stochastic process is defined by the parameters shown in table below.

Description	**Fairmat notation**	**Documentation notation**
$N \times 1$ vector	Y0	Y_0
scalar	delta0	δ_0
$N \times 1$ vector	deltaY	δ_Y
$N \times N$ matrix	K	K
$N \times 1$ vector	theta	θ
$N \times N$ matrix	sigma	Σ
$N \times 1$ vector	alpha	α
$N \times N$ matrix	beta	$\mathcal{B}$
$N \times 1$ vector	lambda1	λ_1
$N \times N$ matrix	lambda2	λ_2
$N \times N$ matrix	I_	I^-

3 Implementation Details

3.1 Introduction on one factor affine term structure models

A short rate model is said to posses an affine term structure if the price of a bond starting at time t and ending at time T is given by

$$p(t,T) = e^{A(t,T)-B(t,T)r(t)}. \quad (1)$$

Assume that the short rate dynamic is

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW(t). \tag{2}$$

It can be demonstrated (see [3]) that if μ and σ are of the form

$$\begin{cases} \mu(t, r) & = & \alpha(t)r + \beta(t) \\ \sigma(t, r) & = & \sqrt{\gamma(t)r + \delta(t)} \end{cases} \tag{3}$$

then the model has an affine term structure where the functions A and B solve the system

$$\begin{cases} \partial_t A(t,T) = \beta(t)B(t,T) - \frac{1}{2}\delta(t)B^2(t,T) \\ A(T,T) = 0 \\ \partial_t B(t,T) = -\alpha(t)B(t,T) + \frac{1}{2}\gamma(t)B^2(t,T) - 1 \\ B(T,T) = 0. \end{cases} \tag{4}$$

In general not all the affine term structure models have a form given by formula (3) but it can be shown that if we make the additional request of having time independent μ and σ, then that's the only possible form for an affine term structure model.

3.2 Dai-Singleton completely affine models

An N-factor affine term structure model is obtained supposing that

1. the short rate is an affine combination of N unobserved variables $Y_1, \ldots, Y_N$

$$r(t) = \delta_0 + \sum_{i=1}^{N} (\delta_Y)_i Y_i(t) = \delta_0 + \delta_Y' Y(t) \tag{5}$$

   where δ_Y and $Y(t)$ are $N \times 1$ vectors

2. the dynamic of $Y(t)$ under the risk-neutral measure Q is given by

$$dY(t) = \tilde{K}(\tilde{\theta} - Y(t))dt + \Sigma\sqrt{S(Y(t))}d\tilde{W}(t) \tag{6}$$

   where

   $S(Y(t))$ is a diagonal matrix with $S_{ii}(Y(t)) = \alpha_i + \beta_i' Y(t)$
   α_i are scalars
   β_i are $N \times 1$ vectors
   $\tilde{W}(t)$ is an N-dimensional independent Wiener process under Q
   $\tilde{K}, \Sigma$ are $N \times N$ matrices
   $\tilde{\theta}$ is an $N \times 1$ vector.

Assuming that the market prices of risk are given by

$$\Lambda(Y(t)) = \sqrt{S(Y(t))}\lambda \tag{7}$$

where λ is an $N \times 1$ vector, the Y dynamic in the physical measure has got the same form as in the risk-neutral measure

$$dY(t) = K(\theta - Y(t))dt + \Sigma\sqrt{S(Y(t))}dW(t) \tag{8}$$

where this time $W(t)$ is an N-dimensional independent Wiener process under P and K, θ are given by

$$K = \tilde{K} - \Sigma\Phi \tag{9}$$

$$\theta = K^{-1}(\tilde{K}\tilde{\theta} + \Sigma\psi) \tag{10}$$

with matrix Φ and vector ψ defined by

$$\Phi = \begin{bmatrix} \lambda_1\beta_i' \\ \vdots \\ \lambda_N\beta_N' \end{bmatrix} \qquad \psi = \begin{bmatrix} \lambda_1\alpha_1 \\ \vdots \\ \lambda_N\alpha_N \end{bmatrix}. \tag{11}$$

Calling α the column vector with components α_i and $\mathcal{B}$ the matrix with columns β_i, a Dai-Singleton model is defined by the set of parameters

$$\{\delta_0, \delta_Y, K, \theta, \Sigma, \alpha, \mathcal{B}, \lambda\}. \tag{12}$$

As explained in [1] this kind of models posses a group of symmetry such that applying to a set of parameters some transformations you obtain another set witch is equivalent to the starting one meaning that the short rate dynamic will be the same in the two cases. So different parameters can represent equivalent models.

3.3 Duffee essentially affine models

In the article [2] Duffee showed how it is possible to extend completely affine models. Starting from the same risk-neutral dynamic as formula (6), he assumes that the market prices of risk are given by

$$\Lambda(Y(t)) = \sqrt{S(Y(t))}\lambda_1 + \sqrt{S^-(Y(t))}\lambda_2 Y(t) \tag{13}$$

where λ_1 is an $N \times 1$ vector, λ_2 is a $N \times N$ matrix and $S^-(Y(t))$ is the diagonal matrix defined as

$$S_{ii}^-(Y(t)) = \begin{cases} S_{ii}^{-1}(Y(t)) = 1/(\alpha_i + \beta_i' Y(t)) & \text{if } \inf\{\alpha_i + \beta_i' Y(t)\} > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

In practice if S_{ii} cannot reach 0, S_{ii}^- is its reciprocal, otherwise $S_{ii}^- = 0$. Note that with $\lambda_2 = 0$ we retrieve the completely affine case.

This definition implies that

- both the risk neutral and the real world dynamics share the same form and are of the same kind as in the completely affine case
- when $S_{ii}(Y(t))$ approaches zero Λ does not go to infinity
- the market prices of risk can change more independently from volatility
- in affine models the instantaneous expected excess return holding a bond between time t and $t+\tau$ is given by

$$e(t,\tau) = -B(\tau)'\Sigma\sqrt{S(Y(t))}\Lambda(Y(t)). \tag{15}$$

  In completely affine models the components of Λ cannot change sign at different t and so given a certain maturity (i.e. fixing the value of τ) also excess returns cannot change sign. Essentially affine models overcome this limitation.

The real world dynamic remains the same but now K is given by

$$K = \tilde{K} - \Sigma\Phi - \Sigma I^{-}\lambda_2 \tag{16}$$

where I^{-} is the diagonal matrix defined by

$$I^{-}_{ii} = 1 \qquad \text{if } S^{-}_{ii}(Y(t)) \neq 0 \tag{17}$$

$$I^{-}_{ii} = 0 \qquad \text{if } S^{-}_{ii}(Y(t)) = 0. \tag{18}$$

An essentially affine model is determined by the set of parameters

$$\{\delta_0, \delta_Y, K, \theta, \Sigma, \alpha, \mathcal{B}, \lambda_1, \lambda_2, I^{-}\} \tag{19}$$

and reduces to the completely affine case if $\lambda_2 = 0_{N\times N}$.

3.4 Admissibility of the model

Depending on $\mathcal{B}$ it's possible that $S_{ii}(t)$ becomes negative for some i at some t. Models for witch this cannot happen are called admissible.

With $\mathcal{B} = 0_{N\times N}$ we have gaussian models that are all admissible. On the other hand with $\mathcal{B} \neq 0_{N\times N}$ the drift and diffusion matrices $K, \theta, \Sigma, \mathcal{B}$ have to satisfy some constrains to assure admissibility. In the article [1] is explained that the higher the rank of $\mathcal{B}$, the more stringent are these constraints and how it is possible to build admissible models.

3.5 Solution to the SDE, expected value and variance

The stochastic differential equation 8 can be solved so that if we know the process at time t_1 the process at time t_2 is given by

$$Y(t_2) = \theta + e^{-K(t_2-t_1)}\left[Y(t_1) - \theta\right] + \int_{t_1}^{t_2} e^{-K(t_2-u)}\Sigma\sqrt{S(Y(u))}dW(u). \tag{20}$$

From this formula it's possible to obtain the conditional expected value and conditional variance for $Y(t_2)$

$$\mathbb{E}[Y(t_2)|Y(t_1)] = \theta + e^{-K(t_2-t_1)}\left[Y(t_1) - \theta\right] \tag{21}$$

$$\mathrm{Var}[Y(t_2)|Y(t_1)] = \int_{t_1}^{t_2} e^{-K(t_2-u)}\Sigma S(\mathbb{E}[Y(u)])S(\mathbb{E}[Y(u)])'\Sigma' e^{-K'(t_2-u)}du. \tag{22}$$

3.6 Bond Price formula

Duffie and Kan demonstrated (see [4]) that in an N-factor affine term structure model the price of a zero coupon bond

$$P(t, t+\tau) = \mathbb{E}_t^Q\left[e^{-\int_t^{t+\tau} r(u)du}\right] \tag{23}$$

is given by

$$P(t, t+\tau) = e^{A(\tau) - B(\tau)'Y(t)} \tag{24}$$

where the functions A and B satisfy the ordinary differential equation

$$\partial_\tau A(\tau) = -\tilde{\theta}'\tilde{K}'B(\tau) + \frac{1}{2}\sum_{i=1}^{N}[\Sigma' B(\tau)]_i^2\alpha_i - \delta_0 \tag{25}$$

$$\partial_\tau B(\tau) = -\tilde{K}'B(\tau) + \frac{1}{2}\sum_{i=1}^{N}[\Sigma' B(\tau)]_i^2\beta_i - \delta_Y \tag{26}$$

with the initial conditions $A(0) = 0, B(0) = 0_{N\times 1}$.

In the DaiSingleton plug-in these differential equations are numerically solved using a fourth-order Runge-Kutta method (for reference see [5]).

Given formula (24) bond's yield are affine in the unobserved factors

$$R(t, t+\tau) = -\frac{1}{\tau}\log\left[P(t, t+\tau)\right] = \frac{1}{\tau}\left[B(\tau)'Y(t) - A(\tau)\right]. \tag{27}$$

3.7 Simulation and discretization scheme

If one tries to simulate an affine term structure model through straight Euler-Maruyama method he obtains that the Y_{n+1} step is created following the formula

$$Y_{n+1} = Y_n + K(\theta - Y_n))\Delta t + \Sigma\sqrt{S(Y(t_n))}\sqrt{\Delta t}N(0,1) \tag{28}$$

where $\Delta t = t_{n+1} - t_n$ and $N(0,1)$ represents a realization of a standard normal random variable. Even if the model is admissible and then from a theoretical point of view the process cannot reach negative values for $S(Y(t))$, its discretized version can generate negative $S(Y(t))$ values.

This forces to choose a different discretization scheme. Fairmat implements the Deelstra-Delbaen discretization scheme, and then the simulation is done through

$$Y_{n+1} = Y_n + K(\theta - Y_n))\Delta t + \Sigma\sqrt{\max\{S(Y(t_n)), 0\}}\sqrt{\Delta t}N(0,1). \tag{29}$$

Indeed negative $S(Y(t))$ can be generated but in this case the diffusion term is suppressed letting the drift term take the process toward values for witch $S(Y(t))$ is positive.

4 Inflation modelling

To price inflation indexed derivatives it's possible to use a particular Dai-Singleton two factor model in which Y_1 e Y_2 are respectively the instantaneous real rate and the instantaneous inflation rate. To better distinguish the two components we will indicate them respectively with ϕ and π. Their sum gives the nominal rate so we can set δ_Y equal to unity vector and $\delta_0 = 0$ to have

$$r(t) = Y_1(t) + Y_2(t) = \phi(t) + \pi(t). \tag{30}$$

Other parameters are set as

- $\alpha_i = 1$
- β_i are all null vectors
- $K = \begin{bmatrix} k_{11} & 0 \\ k_{21} & k_{22} \end{bmatrix}$
- $\Sigma = \begin{bmatrix} \sigma_{11} & 0 \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$
- θ is a 2×1-vector

With this choice of parameters the differential equation (25) and (26) can be analytically solved and the bond function is given by

$$P(t, t+\tau) = e^{A(\tau) - B_\phi(\tau)\phi(t) - B_\pi(\tau)\pi(t)} \tag{31}$$

5 Calibration

The plug-in can calibrate the model in two ways, one in which the process components represent principal components of the zero rate curve and the other in which the process components are the instantaneous real rate and instantaneous inflation rate as in section 4.

5.1 Principal component analysis calibration

In this calibration case the model is a two factor Gaussian model where Y_1 and Y_2 represents the first two principal components obtained through PCA over an historical series of zero rate curve.

Calibration is carried out in two steps. The first one consists in finding the parameter subset which better reflects the descriptive statistics of principal components. In the second step, assuming a Gaussian distribution for the noise, the historical series of latent components can be inferred through a Kalman filter analysis and then the remaining parameters can be fixed using maximum likelihood estimation applied to forecast errors on the historical series.

5.2 Inflation calibration

In this calibration case the model is a two factor Gaussian model where $Y_1 = \phi$ and $Y_2 = \pi$ components represent respectively the instantaneous real rate and the instantaneous inflation rate as in section 4.

Calibration is based on historical series of nominal zero rate curve and historical series of inflation term structure calculated from quoted zero coupon inflation indexed swap (ZCIIS).

We will indicate with Υ the nominal interest rate matrix and with Π the inflation rate matrix, with rows representing rate maturities and columns representing different observation.

Using this notation we can write the following system

$$\begin{cases} \Upsilon(\tau) = -\dfrac{A(\tau)}{\tau} + \dfrac{B_\phi(\tau)}{\tau}\phi + \dfrac{B_\pi(\tau)}{\tau}\pi + \epsilon(\tau) \\ \\ \Pi(\tau) = -\dfrac{A_\pi(\tau)}{\tau} + \dfrac{B_\pi(\tau)}{\tau}\pi + u(\tau) \end{cases} \tag{32}$$

and assume that

$$\begin{aligned} \mathbb{E}\Big[\epsilon(t)\epsilon'(s)\Big] &= \begin{cases} \sigma_y^2 & t = s \\ 0 & t \neq s \end{cases} \\ \mathbb{E}\Big[u(t)u'(s)\Big] &= \begin{cases} \sigma_x^2 & t = s \\ 0 & t \neq s \end{cases} \\ \mathbb{E}\Big[\epsilon(t)u'(s)\Big] &= \quad 0 \quad \forall t, s \quad . \end{aligned}$$

Through equations (32) and assuming Gaussian noise distribution latent variables can be calculated using Kalman filter analysis. Then process parameters can be inferred using maximum likelihood estimation applied to forecast errors on historical series.

References

[1] Qiang Dai and Kenneth Singleton. Specification analysis of affine term structure models. *The Journal of Finance*, LV(5):1943–1978, oct 2000.

[2] Gregory R. Duffee. Term premia and interest rate forecasts in affine models. *Journal of Finance*, LVII(1):405–443, 2002.

[3] Tomas Björk. *Arbitrage theory in continuous time*. Oxford University Press, 2004.

[4] Darrel J. Duffie and Rui Kan. A yield-factor model of interest rates. *Mathematical Finance*, 6:379–406, 1996.

[5] S. D. Conte and Carl de Boor. *Elementary numerical analysis*. McGraw-Hill, 5th ed. edition, 1980.